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Nonlinear balanced realization based on singular value analysis of Hankel operators

Kenji Fujimoto[a] and Jacquelien M. A. Scherpen[b]

[a]Department of Systems Science
Graduate School of Informatics
Kyoto University
Uji, Kyoto 611-0011, Japan
fuji@robot.kuass.kyoto-u.ac.jp

[b]Faculty Information Technology & Systems
Delft University of Technology
P.O. Box 5031, 2600 GA Delft
The Netherlands
J.M.A.Scherpen@dcsc.tudelft.nl

***Abstract*—In this paper a new balanced realization method for nonlinear systems is proposed which is based on singular value analysis of Hankel operators. The proposed method balances the relationship between the input-to-state behavior and the state-to-output behavior of nonlinear dynamical systems, whereas the existing results only balance the relationship among the coordinate axes of the state-space. This result is expected to be a basis for new model reduction and system identification of nonlinear systems.**

I. Introduction

The nonlinear extension of the state-space concept of balanced realizations has been introduced in [10], mainly based on studying the past input energy and the future output energy. Since then, many results on state-space balancing, modifications, computational issues for model reduction and related minimality considerations for nonlinear systems have appeared in the literature, e.g. [4], [5], [8], [9], [11]. In particular, *singular value functions* which are nonlinear state-space extension of the Hankel singular values in the linear case play an important role in the nonlinear Hankel theory. However, the original characterization in [10] was incomplete in the sense that they are not unique and the resulting model reduction procedure gives different reduced models according to the choice of different set of singular value functions.

The authors proposed a new characterization of Hankel singular value functions which have closer relationship to the gain structure of the Hankel operator in [1]. The new singular value functions are called *axis singular value functions* and are characterized by *singular value analysis* of the Hankel operators. Although their original definition has no relationship with the conventional singular functions, it was recently shown that the new and conventional singular value functions coincide with each other when the system has a special state-space realization, which can always be obtained by a coordinate transformation. In [2], this special state-space realization was adopted as the new characterization of input-normal/output-diagonal balanced realization. It was also proved that model reduction based on this state-space realization preserves several important properties of the original system such as the Hankel norm, controllability and observability properties and so on. However, the above balancing procedure only gives balance among the coordinate axes of the state-space. On the other hand, the balanced realization of the linear systems also balances the relationship between the input-to-state behavior and the state-to-output behavior. From numerical point of view, this property was quite important in the linear case.

The main objective of this paper is to establish the balanced realization which balances both of the relationship among the coordinate axes of the state-space and that between the input-to-state behavior and the state-to-output behavior. This realization is derived based on the techniques developed in [1], [2], [3] and strongly depends on the input-normal/output-diagonal balanced realization based on singular value analysis of Hankel operators. The authors believe that the proposed results will provide a new basis for model reduction and system identification of nonlinear dynamical systems.

II. Linear systems as a paradigm

This section gives some examples of linear balancing theory which plays an important role in the model reduction and identification of linear systems, see e.g. [12]. We present them here in a way that clarifies the line of thinking in the nonlinear case. Consider a causal linear input-output system $\Sigma : L_2^m[0,\infty) \to L_2^r[0,\infty)$ with a state-space realization

$$u \mapsto y = \Sigma(u) : \left\{ \begin{array}{rcl} \dot{x} & = & Ax + Bu \\ y & = & Cx \end{array} \right. \qquad (1)$$

where $x(0) = 0$. Its Hankel operator is given by the composition of its observability and controllability operators $\mathcal{H} = \mathcal{O} \circ \mathcal{C}$, where the observability and controllability operators, $\mathcal{O} : \mathbb{R}^n \to L_2^r[0,\infty)$ and $\mathcal{C} : L_2^m[0,\infty) \to \mathbb{R}^n$, respectively, are given by

$$\begin{array}{rcl} x^0 \mapsto y = \mathcal{O}(x^0) & := & Ce^{At}x^0 \\ u \mapsto x^0 = \mathcal{C}(u) & := & \displaystyle\int_0^\infty e^{A\tau} Bu(\tau)\, d\tau. \end{array}$$

The Hankel, controllability and observability operators are closely related to the observability and controllability Gramians by $Q = \mathcal{O}^* \circ \mathcal{O}$ and $P = \mathcal{C} \circ \mathcal{C}^*$. Furthermore, from e.g. Theorem 8.1 in [12], we know the following property.

Theorem 1: [12] The operator $\mathcal{H}^* \circ \mathcal{H}$ and the matrix QP have the same nonzero eigenvalues.

The square roots of the eigenvalues of QP are called the Hankel singular values of the system (1) and are denoted by σ_i's where $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_n$. In fact, the largest singular value characterizes the Hankel norm $\|\Sigma\|_H$ of the system Σ

$$\|\Sigma\|_H := \sup_{\substack{u \in L_2^+ \\ u \neq 0}} \frac{\|\mathcal{H}(u)\|_{L_2}}{\|u\|_{L_2}} = \sigma_1. \tag{2}$$

Further, using a similarity transformation (linear coordinate transformation), we can diagonalize both P and Q and furthermore let them coincide with each other, i.e.,

$$P = Q = \mathrm{diag}(\sigma_1, \sigma_2, \ldots, \sigma_n). \tag{3}$$

This state-space realization is called *balanced realization.* The system is balanced in two senses: (i) P and Q are in a diagonal form which means that the relationship between the coordinate axes of the state-space is balanced in terms of Hankel singular values, and (ii) $P = Q$ which means that the relationship between the input-to-state behavior and the state-to-output behavior is balanced.

III. Preliminaries

The balanced realization and the related techniques for nonlinear system have been developed along the way of thinking described in the previous section. First of all, as generalization of controllability and observability Gramians, controllability and observability functions of nonlinear systems were introduced in [10]. Consider an input-affine nonlinear system Σ

$$u \mapsto y = \Sigma(u) : \left\{ \begin{array}{rcl} \dot{x} & = & f(x) + g(x)u \\ y & = & h(x) \end{array} \right. \tag{4}$$

with $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$ and $y(t) \in \mathbb{R}^r$. Then its controllability function $L_c(x)$ and observability function $L_o(x)$ are defined by

$$\begin{array}{rcl} L_c(\xi) & := & \displaystyle\inf_{\substack{u \in L_2^- \\ x(-\infty)=0, x(0)=\xi}} \frac{1}{2}\|u\|_{L_2}^2 \\ L_o(\xi) & := & \displaystyle\frac{1}{2}\|y\|_{L_2}^2, \ x(0) = \xi, \ u = 0 \end{array}$$

In the linear case,

$$L_c(x) = \frac{1}{2}x^{\mathrm{T}}P^{-1}x, \ \ L_o(x) = \frac{1}{2}x^{\mathrm{T}}Qx$$

hold with the controllability and observability Gramians P and Q. The first balancing theory was given as follows.

Theorem 2: [10] Consider the operator Σ with the asymptotically stable state-space realization (4). Then there exists a neighborhood U of the origin and a smooth coordinate transformation $x = \Phi(z)$ on U converting Σ into an input-normal/output-diagonal form, where

$$\begin{array}{rcl} L_c(\Phi(z)) & = & \displaystyle\frac{1}{2}\, z^{\mathrm{T}} z \end{array} \tag{5}$$

$$\begin{array}{rcl} L_o(\Phi(z)) & = & \displaystyle\frac{1}{2}\sum_{i=1}^{n} z_i^2 \tau_i(z) \end{array} \tag{6}$$

with $\tau_1(z) \geq \ldots \geq \tau_n(z)$ being the so called smooth singular value functions on U.

The *input-normal/output-diagonal* realization is a basis of what follows. However, this result is incomplete in the sense that the properties (i) and (ii) explained below the equation (3) are not fulfilled exactly. Indeed this realization is not unique [4] and, consequently, the corresponding model reduction procedure gives different reduced models according to the choices of different sets of singular value functions. Recently, some developments on the balanced realization have been done which achieves *unique* and more precise characterization of input-normal/output-diagonal realization. Also it has a closer relationship to the nonlinear version of the Hankel operator. As in the linear case, the Hankel operator $\mathcal{H}$ of the system Σ in (4) is given by the composition of the observability and controllability operators $\mathcal{H} = \mathcal{O} \circ \mathcal{C}$ where

$$y = \mathcal{O}(x^0) : \left\{ \begin{array}{rcl} \dot{x} & = & f(x) \quad x(0) = x^0 \\ y & = & h(x) \end{array} \right. \tag{7}$$

$$x^1 = \mathcal{C}(u) \ : \left\{ \begin{array}{rcl} \dot{x} & = & f(x) + g(x)\mathcal{F}_-(u) \quad x(-\infty) = 0 \\ x^1 & = & x(0) \end{array} \right. \tag{8}$$

Here $\mathcal{F}_- : L_2^m[0, \infty) \rightarrow L_2^m(-\infty, \infty)$ is the time flipping operator defined by

$$\hat{u} \mapsto u = \mathcal{F}_-(\hat{u}) := \left\{ \begin{array}{ccc} \hat{u}(-t) & : & t < 0 \\ 0 & : & t \geq 0 \end{array} \right. .$$

Instead of considering the eigenstructure of $\mathcal{H}^* \circ \mathcal{H}$ as in Theorem 1, the solution pair $\lambda \in \mathbb{R}$ and $v \in L_2^+$ of

$$(\mathrm{d}\mathcal{H}(v))^* \circ \mathcal{H}(v) = \lambda \, v$$

is considered. Investigating its solution is called *singular value analysis* of $\mathcal{H}$. In the authors' former result [1], it was proved that there exist n independent solution curves of the form

$$\begin{array}{rcl} \lambda & = & \lambda_i(s) \\ v & = & v_i(s) \ \ , \ \ i = 1, 2, \ldots, n, \ \ s \in \mathbb{R} \\ \|v\|_{L_2} & = & |s| \end{array}$$

which are parameterized by s. The related input-output ratio of the Hankel operator defined by

$$\begin{array}{rcl} \rho_i(s) & := & \displaystyle\frac{\|\mathcal{H}(v_i(s))\|_{L_2}}{\|v_i(s)\|_{L_2}} \\ \min\{\rho_i(s), \rho_i(-s)\} & > & \max\{\rho_{i+1}(s), \rho_{i+1}(-s)\} \end{array}$$

are called *axis singular value functions.* They have a closer relationship (than conventional singular value functions τ_i's) to the Hankel operator because it satisfies

$$\|\Sigma\|_H = \sup_{s \in \mathbb{R}} \rho_1(s)$$

in a similar way to the linear case (2). Also ρ_i's are uniquely determined since they are defined only using the input-output property of the Hankel operator.

Furthermore, in [2], [3], it was shown that there exists an input-normal form whose singular value functions τ_i's have a close relationship to the axis singular value functions ρ_i's defined above. The result was proved under the following assumptions.

Assumption A1 Suppose that the system Σ in (4) is asymptotically stable about the origin, that there exist a neighborhoods of the origin where the operators $\mathcal{O}$, $\mathcal{C}$ and $\mathcal{C}^\dagger$ exist and are smooth. Here $\mathcal{C}^\dagger$ denotes the pseudo-inverse of $\mathcal{C}$.

Assumption A2 Suppose that the Hankel singular values of the Jacobian linearization of the system Σ are nonzero and distinct.

Theorem 3: [2] Consider the operator Σ with the state-space realization (4). Suppose that Assumptions A1 and A2 hold. Then there exist a neighborhood U of 0 and a coordinate transformation $x = \Phi(z)$ on U converting the system an input-normal form (5) and (6) satisfying the following properties.

$$z_i = 0 \quad \Leftrightarrow \quad \frac{\partial L_c(\Phi(z))}{\partial z_i} = 0 \quad \Leftrightarrow \quad \frac{\partial L_o(\Phi(z))}{\partial z_i} = 0$$

holds for all $i \in \{1, 2, \ldots, n\}$ on U. Furthermore

$$\tau_i(0, \ldots, 0, \underbrace{z_i}_{i\text{-th}}, 0, \ldots, 0) = \rho_i(z_i)^2$$

$$\frac{\partial \tau_i}{\partial z}(0, \ldots, 0, \underbrace{z_i}_{i\text{-th}}, 0, \ldots, 0) = (0, \ldots, 0, \underbrace{\frac{d\rho_i(z_i)^2}{dz_i}}_{i\text{-th}}, 0, \ldots, 0)$$

holds for all $i \in \{1, 2, \ldots, n\}$. In particular, if $U = \mathbb{R}^n$, then

$$\|\Sigma\|_H^2 = \sup_{z_1 \in \mathbb{R}} \tau_1(z_1, 0, \ldots, 0).$$

By this theorem, we can obtain an input-normal/output-diagonal realization which has a close relationship to the Hankel operator and which is almost uniquely determined was obtained. In fact, this theorem gives the nonlinear version of the property (i) explained below the equation (3). Furthermore, the corresponding model reduction procedure gives unique reduced order models [2]. However, the nonlinear version of the characterization (ii) was not obtained so far. This is the main topic of the remainder of the present paper.

IV. Main results

The main contribution obtained here is the balancing between the controllability and observability functions which gives much clearer relationships between the input-to-state and state-to-output behavior of nonlinear dynamical systems. In order to prove the general case, let us consider the 2-dimensional case at first, which plays the key role in the proof of the general case result.

Lemma 1: Consider the state-space realization (4) with the dimension $n = 2$. Suppose that Assumptions A1 and A2 hold. Then there exist a neighborhood U of the origin and a coordinate transformation $x = \Phi(z)$ on U converting the system into the following form

$$\begin{aligned} L_c(\Phi(z)) &= \frac{1}{2}\left(z_1^2 + z_2^2\right) \\ L_o(\Phi(z)) &= \frac{1}{2}\left((z_1\rho_1(z_1))^2 + (z_2\rho_2(z_2))^2\right). \end{aligned}$$

Proof: It is assumed without loss of generality that the system is already balanced in the sense of Theorem 3 on the coordinate x, that is,

$$L_c(x) = \frac{1}{2}\left(x_1^2 + x_2^2\right) \tag{9}$$

$$L_o(x) = \frac{1}{2}\left((x_1\tau_1(x))^2 + (x_2\tau_2(x))^2\right) \tag{10}$$

$$\rho_1(x_1)^2 = \tau_1(x_1, 0) \tag{11}$$

$$\rho_2(x_2)^2 = \tau_2(0, x_2) \tag{12}$$

$$x_i = 0 \iff \frac{\partial L_o}{\partial x_i} = 0. \tag{13}$$

Let $\tilde{L}_o(z)$ denote the balanced observability function, that is,

$$\tilde{L}_o(z) := \frac{1}{2}\left((z_1\rho_1(z_1))^2 + (z_2\rho_2(z_2))^2\right).$$

The coordinate transformation $x = \Phi(z)$ has to be the solution of the pair of equations

$$F_c(x, z) := L_c(x) - L_c(z) = 0 \tag{14}$$

$$F_o(x, z) := L_o(x) - \tilde{L}_o(z) = 0. \tag{15}$$

Define the polar coordinates

$$\theta := \begin{pmatrix} r \\ \theta_1 \end{pmatrix} = \begin{pmatrix} \sqrt{x_1^2 + x_2^2} \\ \operatorname{atan2}(x_2, x_1) \end{pmatrix} =: \Theta(x)$$

$$\varphi := \begin{pmatrix} s \\ \varphi_1 \end{pmatrix} = \begin{pmatrix} \sqrt{z_1^2 + z_2^2} \\ \operatorname{atan2}(z_2, z_1) \end{pmatrix} = \Theta(z).$$

Note that the equation (14) is satisfied if and only if $s = r$, that is,

$$F_c(\Theta^{-1}(r, \theta), \Theta^{-1}(r, \varphi)) \equiv 0$$

holds. Hence, what we have to solve is (15), namely

$$0 = F_o(\Theta^{-1}(r, \theta_1), \Theta^{-1}(r, \varphi_1)).$$

The derivative of F_o can be calculated as

$$\begin{aligned} \frac{\partial F_o}{\partial \theta_1} &= \frac{\partial F_o(x,z)}{\partial x}\frac{\partial \Theta^{-1}(\theta)}{\partial \theta_1} = \frac{\partial L_o(x)}{\partial x}\frac{\partial \Theta^{-1}(\theta)}{\partial \theta_1} \\ &= -\frac{\partial L_o(x)}{\partial x_1} r \sin\theta_1 + \frac{\partial L_o(x)}{\partial x_2} r \cos\theta_1 \\ &= -\frac{\partial L_o(x)}{\partial x_1} x_2 + \frac{\partial L_o(x)}{\partial x_2} x_1 \end{aligned} \tag{16}$$

$$\frac{\partial F_o}{\partial \varphi_1} = \frac{\partial \tilde{L}_o(z)}{\partial z_1} z_2 - \frac{\partial \tilde{L}_o(z)}{\partial z_2} z_1. \tag{17}$$

The relationship (13) and the Morse's lemma (e.g. Lemma 2.1 in [6]) imply that there exist smooth scalar functions $\ell_i(x)$'s and $\tilde{\ell}_i(z_i)$'s satisfying

$$\frac{\partial L_o(x)}{\partial x_i} = x_i \ell_i(x), \quad \frac{\partial \tilde{L}_o(z)}{\partial z_i} = z_i \tilde{\ell}_i(z_i)$$

which reduce (16) and (17) into

$$\frac{\partial F_o}{\partial \theta_1} = -x_1 x_2(\ell_1 - \ell_2), \quad \frac{\partial F_o}{\partial \varphi_1} = z_1 z_2(\tilde{\ell}_1 - \tilde{\ell}_2). \tag{18}$$

The functions ℓ_i's and $\tilde{\ell}_i$'s coincide with the Hankel singular values σ_i's of the Jacobian linearization of the system at the origin, i.e.,

$$\ell_i(0) = \tilde{\ell}_i(0) = \sigma_i.$$

Assumption A2 guarantees that there exists a neighborhood of the origin where $\ell_1(x) > \ell_2(x)$, $\tilde{\ell}_1(z_1) > \tilde{\ell}_2(z_2)$ hold. Hence the equations (18) and (18) imply

$$\frac{\partial F_o}{\partial \theta_1} = 0 \iff x_1 x_2 = 0 \iff \theta_1 = 0 \bmod \frac{\pi}{2} \tag{19}$$

$$\frac{\partial F_o}{\partial \varphi_1} = 0 \iff z_1 z_2 = 0 \iff \varphi_1 = 0 \bmod \frac{\pi}{2} \tag{20}$$

hold in the neighborhood of the origin. On the other hand, the equations (10), (11) and (12) imply

$$\theta_1 = 0 \bmod \frac{\pi}{2} \;\Rightarrow\; L_o(x) = \tilde{L}_o(x) \;\Rightarrow\; F_o(x,x) = 0.$$

That is, the coordinate transformation $x = \Phi(z)$ has to coincide with the identity on the axes $x = (x_1, 0)$ and $x = (0, x_2)$. For any $\theta_1 \neq 0 \bmod \pi/2$ and any $r \neq 0$, the intermediate value theorem suggests that there exists a corresponding φ_1 and vice versa. Therefore we can define a one-to-one mapping (for an arbitrary fixed r) between θ_1 and φ_1 by $\theta_1 = \psi(r, \varphi_1)$. Using this function ψ, we can construct a one-to-one mapping between θ and φ

$$\theta = \begin{pmatrix} r \\ \theta_1 \end{pmatrix} = \begin{pmatrix} s \\ \psi(s, \varphi_1) \end{pmatrix} =: \Psi(\varphi)$$

Furthermore, the implicit function theorem and the relationships (19) and (20) imply that the mapping $\varphi_1 \mapsto \theta_1$ is a diffeomorphism at least for all $\varphi_1 \neq 0 \bmod \pi/2$. Hence what we have to prove in the rest is the smoothness of $\Phi(z)$ at the points where $\varphi_1 = 0 \bmod \pi/2$. The Jacobian matrix of Ψ is given by

$$\frac{\partial \Psi(\varphi)}{\partial \varphi} = \begin{pmatrix} 1 & 0 \\ \frac{\partial \theta_1}{\partial s} & \frac{\partial \theta_1}{\partial \varphi_1} \end{pmatrix}$$

Here the implicit function theorem also implies

$$\begin{aligned} \frac{\partial \theta_1}{\partial \varphi_1} &= -\frac{\frac{\partial F_o}{\partial \varphi_1}}{\frac{\partial F_o}{\partial \theta_1}} = \frac{z_1 z_2(\tilde{\ell}_1(z_1) - \tilde{\ell}_2(z_2))}{x_1 x_2(\ell_1(x) - \ell_2(x))} \\ &= \frac{\sin\varphi_1 \cos\varphi_1(\tilde{\ell}_1(z_1) - \tilde{\ell}_2(z_2))}{\sin\theta_1 \cos\theta_1(\ell_1(x) - \ell_2(x))}. \end{aligned}$$

In order to investigate the relationship between θ_1 and φ_1 around the point $\theta_1 = \varphi_1 = 0$, let us consider the limit of the above relationship as $\theta_1 \to 0$ and $\varphi_1 \to 0$ by considering r as a constant

$$\begin{aligned} &((\ell_1(r,0) - \ell_2(r,0))\theta_1 + o(\theta_1))\,\mathrm{d}\theta_1 \\ &= \left((\tilde{\ell}_1(r) - \tilde{\ell}_2(0))\varphi_1 + o(\varphi_1)\right)\mathrm{d}\varphi_1. \end{aligned}$$

Integrating this equation using the continuity of $\theta_1 = \psi(\varphi_1)$ yields

$$\theta_1^2 + o(\theta_1^2) = \tfrac{\tilde{\ell}_1(r) - \tilde{\ell}_2(0)}{\ell_1(r,0) - \ell_2(r,0)}\varphi_1^2 + o(\varphi_1^2)$$

which implies

$$\theta_1 = \sqrt{\tfrac{\tilde{\ell}_1(r) - \tilde{\ell}_2(0)}{\ell_1(r,0) - \ell_2(r,0)}}\varphi_1 + o(\varphi_1)$$

Therefore the mapping $\theta_1 = \psi(s, \varphi_1)$ is smooth which suggests that $\theta = \Psi(\varphi)$ is also smooth. Similar relationships hold in the other cases $\theta_1 = \varphi_1 = \pm(\pi/2), \pi$. This completes the proof. ∎

Using this lemma recursively and repeatedly, our main result can be obtained, where all the coordinate axes of the state-pace appear separately in the observability and controllability functions.

Theorem 4: Consider the operator Σ with the state-space realization (4). Suppose that Assumptions A1 and A2 hold. Then there exist a neighborhood U of the origin and a coordinate transformation $x = \Phi(z)$ on U converting the system into the following form

$$L_c(\Phi(z)) = \frac{1}{2} z^{\mathrm{T}} z \tag{21}$$

$$L_o(\Phi(z)) = \frac{1}{2}\sum_{i=1}^{n} (z_i \rho_i(z_i))^2. \tag{22}$$

Proof: As in the proof of Lemma 1, it is assumed without loss of generality that the system is already balanced in the sense of Theorem 3 on the coordinate x. The theorem is proved by induction with respect to the dimension n.

(i) Case $n = 1$ holds obviously.

(ii) Case $n = 2$ is proved in Lemma 1.

(iii) Case $n = k$**:** Suppose that the theorem holds in the case $n = k - 1$. Let us define truncated vectors $\check{(\cdot)}_i$ and $\check{(\cdot)}_i$ for a given vector $x = (x_1, x_2, \ldots, x_k) \in \mathbb{R}^k$ by

$$\begin{aligned} \check{x}_i &:= (x_1, \ldots, x_{i-1}, 0, x_{i+1}, \ldots, x_k) \in \mathbb{R}^k \\ \check{x}_i &:= (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_k) \in \mathbb{R}^{k-1}. \end{aligned}$$

First of all, let us apply the theorem to the system restricted to the subspace $\{x \mid x_k = 0\}$. The theorem in the case $n = k - 1$ (assumed above) implies that there exists a coordinate transformation $\check{x}_k = \Psi_k(\check{z}_k)$ satisfying

$$L_c(\Psi_k(\check{z}_k), 0) = \tfrac{1}{2}\check{z}_k^{\mathrm{T}}\check{z}_k, \quad L_o(\Psi_k(\check{z}_k), 0) = \tfrac{1}{2}\textstyle\sum_{i=1}^{k-1}(z_i\rho_i(z_i))^2.$$

As in the proof of Lemma 1, in order to construct a coordinate transformation preserving the input-normal form, let us define the generalized polar coordinate

$$\theta := \begin{pmatrix} r \\ \theta_1 \\ \vdots \\ \theta_{k-1} \end{pmatrix} = \begin{pmatrix} (x_1^2 + x_2^2 + \cdots + x_n^2)^{1/2} \\ \mathrm{atan2}(x_2, x_1) \\ \vdots \\ \mathrm{atan2}(x_k, (x_1^2 + \cdots + x_{k-1}^2)^{1/2}) \end{pmatrix} =: \Theta(x).$$

By definition, $x_1 = 0 \Leftrightarrow \theta_1 = \pi/2$ and $x_{i+1} = 0 \Leftrightarrow \theta_i = 0$. We also define the generalized polar coordinate corresponding to z by $\varphi := (s, \varphi_1, \ldots, \varphi_{k-1}) := \Theta(z)$. Then the function Ψ_k has to satisfy

$$\check{\theta}_k = \check{\Theta}_k \circ \Psi_k \circ \check{\Theta}_k^{-1}(\check{\varphi}_k) =: \tilde{\Psi}_k(\check{\varphi}_k). \tag{23}$$

On these coordinates, consider a rotational matrix $R(\varphi, \theta) \in \mathbb{R}^{k\times k}$ changing the polar coordinate θ into φ with $s = r$ defined by $R(\varphi, \theta) := R_{k-1}(\varphi_{k-1}) \ldots R_1(\varphi_1) R_1(-\theta_1) \ldots R_{k-1}(-\theta_{k-1})$ with $R_i(\theta_i)$'s the rotation matrices for the component angles θ_i, $i = 1, \ldots, k-1$ defined by

$$R_1(\theta_1) := \begin{pmatrix} 1 & 0 & 0 & & 0 \\ 0 & \cos\theta_1 & -\sin\theta_1 & & \\ 0 & \sin\theta_1 & \cos\theta_1 & & \\ & & & 1 & 0 \\ & & & & \ddots \\ 0 & & & 0 & 1 \end{pmatrix} \in \mathbb{R}^{k\times k}$$

$$R_2(\theta_2) := \begin{pmatrix} 1 & 0 & 0 & 0 & & 0 \\ 0 & \cos\theta_2 & 0 & -\sin\theta_2 & & \\ 0 & 0 & 1 & 0 & & \\ 0 & \sin\theta_2 & 0 & \cos\theta_2 & & \\ & & & & 1 & 0 \\ & & & & & \ddots \\ 0 & & & & 0 & 1 \end{pmatrix} \in \mathbb{R}^{k\times k}$$

$$\vdots$$

Using the mapping $\check{\varphi}_k = \tilde{\Psi}_k^{-1}(\check{\theta}_k)$ defined on S^{k-1}, we can construct a coordinate transformation on S^k by

$$\begin{aligned} \varphi &= \Psi_k^{-1}(\theta) \\ &:= (\lambda(\theta_k)\tilde{\Psi}_k^{-1}(\check{\theta}_k) + (1 - \lambda(\theta_k))\check{\theta}_k, \theta_k) \end{aligned}$$

where λ is a smooth scalar function with an appropriate constant ϵ $(0 < \epsilon < \pi/2)$, c.f. [7]

$$\lambda(s) := \begin{cases} 0 & (s \geq 2\epsilon) \\ \frac{\exp(\epsilon/s)}{\exp(-\epsilon/(s+\epsilon)) + \exp(\epsilon/s)} & (\epsilon \leq s \leq 2\epsilon) \\ 1 & (-\epsilon \leq s \leq \epsilon) \\ \frac{\exp(-\epsilon/(s+2\epsilon))}{\exp(-\epsilon/(s+2\epsilon)) + \exp(\epsilon/(s+\epsilon))} & (-2\epsilon \leq s \leq -\epsilon) \\ 0 & (s \leq -2\epsilon) \end{cases}.$$

It is readily observed that $\Psi_k(\check{\varphi}_k) = \tilde{\Psi}_k(\check{\varphi}_k)$. Furthermore, a coordinate transformation on $\mathbb{R}^k$ can be constructed by

$$x = \Phi_k(\xi) := R(\Psi_k(\varphi), \varphi)\xi = R(\Psi_k \circ \Theta(\xi), \Theta(\xi))\xi$$

which is defined on a neighborhood of the origin. By its construction this coordinate transformation $x = \Phi_k(\xi)$ satisfies

$$L_o(\Phi_k(\check{\xi}_k)) = \tfrac{1}{2}\textstyle\sum_{i=1}^{k-1}(\xi_i \rho_i(\xi_i))^2 \tag{24}$$

without losing the properties achieved in Theorem 3.

Next let us construct a coordinate transformation $\xi = \Phi_{k-1}(\zeta)$ which achieves the balanced realization in the subspace $\{\xi \mid \xi_{k-1} = 0\}$, that is,

$$L_o(\Phi_k \circ \Phi_{k-1}(\check{\zeta}_{k-1})) = \tfrac{1}{2}\textstyle\sum_{i=1, i\neq k-1}^{k}(\zeta_i \rho_i(\zeta_i))^2. \tag{25}$$

Since the subspace $\{\xi \mid \xi_{k-1} = \xi_k = 0\}$ is already balanced in the sense that (24) already holds, Φ_k can be chosen in such a way that it coincides with the identity on $\{\zeta \mid \zeta_{k-1} = \zeta_k = 0\}$. This fact reveals that the following property also holds.

$$L_o(\Phi_k \circ \Phi_{k-1}(\check{\zeta}_k)) = \tfrac{1}{2}\textstyle\sum_{i=1}^{k-1}(\zeta_i \rho_i(\zeta_i))^2 \tag{26}$$

Furthermore, since the coordinate transformations constructed here preserves the properties in Theorem 3, we have

$$\begin{aligned} L_c(\Phi_k \circ \Phi_{k-1}(\zeta)) &= \tfrac{1}{2}\zeta^{\mathrm{T}}\zeta & (27) \\ L_o(\Phi_k \circ \Phi_{k-1}(\zeta)) &= \tfrac{1}{2}\textstyle\sum_{i=1}^{k}(\zeta_i \bar{\tau}_i(\zeta))^2 & (28) \\ \rho_i(\zeta_i)^2 &= \bar{\tau}_i(0, \ldots, 0, \underbrace{\zeta_i}_{i\text{-th}}, 0, \ldots, 0) & (29) \\ \zeta_i = 0 &\iff \tfrac{\partial L_o(\Phi_k \circ \Phi_{k-1}(\zeta))}{\partial \zeta_i} = 0. & (30) \end{aligned}$$

Now let us define a virtual controllability and observability functions of ζ_{k-1} and ζ_k by regarding the other variables ζ_i's $(i = 1, 2, \ldots, k-2)$ as constants

$$\begin{aligned} \bar{L}_c(\zeta_{k-1}, \zeta_k) &:= \tfrac{1}{2}(\zeta_{k-1}^2 + \zeta_k^2) \\ \bar{L}_o(\zeta_{k-1}, \zeta_k) &:= L_o(\Phi_k \circ \Phi_{k-1}(\zeta)) - \tfrac{1}{2}\textstyle\sum_{i=1}^{k-2}(\zeta_i \rho_i(\zeta_i))^2. \end{aligned}$$

Note that, due to the relationships (25), (26) and (30), this function satisfies the following properties at least in a neighborhood of the origin for any ζ_i's $(i = 1, 2, \ldots, k-2)$.

$$\begin{aligned} &\bar{L}_o(\zeta_{k-1}, \zeta_k) \geq 0 \\ &\bar{L}_o(\zeta_{k-1}, \zeta_k) = 0 \iff \zeta_{k-1} = \zeta_k = 0 \end{aligned}$$

The properties (27)–(30) implies that these functions are already balanced in the sense of Theorem 3. Therefore, application of Lemma 1 to this pair of functions on the state-space (ζ_{k-1}, ζ_k) proves the existence of a coordinate transformation $(\zeta_{k-1}, \zeta_k) = \bar{\phi}(\bar{\zeta}_{k-1}, \bar{\zeta}_k)$ (which also depends on ζ_i's $(i = 1, 2, \ldots, k-2)$) satisfying

$$\begin{aligned} \bar{L}_c(\bar{\phi}(\bar{\zeta}_{k-1}, \bar{\zeta}_k)) &= \tfrac{1}{2}(\bar{\zeta}_{k-1}^2 + \bar{\zeta}_k^2) \\ \bar{L}_o(\bar{\phi}(\bar{\zeta}_{k-1}, \bar{\zeta}_k)) &= \tfrac{1}{2}((\bar{\zeta}_{k-1}\rho_{k-1}(\bar{\zeta}_{k-1}))^2 + (\bar{\zeta}_k \rho_k(\bar{\zeta}_k))^2). \end{aligned}$$

Let us define a coordinate transformation on $\mathbb{R}^k$ by

$$\begin{aligned} x &= \Phi(z) := \Phi_k \circ \Phi_{k-1} \circ \bar{\Phi}(z) \\ \bar{\Phi}(z) &:= \begin{pmatrix} z_1 \\ \vdots \\ z_{k-2} \\ \bar{\phi}_1(z_{k-1}, z_k; z_1, \ldots, z_{k-2}) \\ \bar{\phi}_2(z_{k-1}, z_k; z_1, \ldots, z_{k-2}) \end{pmatrix} \end{aligned}$$

where $\bar{\phi} = (\bar{\phi}_1, \bar{\phi}_2)$ and its arguments $(z_{k-1}, z_k; z_1, \ldots, z_{k-2})$ explicitly describe its dependency on the variables $z_1, \ldots, z_{k-2}$. It can be observed that the properties (21) and (22) hold on the coordinate z obtained here.

Finally, the cases (i), (ii) and (iii) prove the theorem by induction. ■

Once we obtain the observability and controllability functions which are separated into n functions according to the coordinate axes, it is easy to obtain the *real* balanced realization including balancing between the input-to-state behavior and the state-to-output behavior.

Theorem 5: Consider the operator Σ with the state-space realization (4). Suppose that Assumptions A1 and A2 hold. Then there exist a neighborhood U of the origin and a coordinate transformation $x = \Phi(z)$ on U converting the system into the following form

$$\begin{aligned} L_c(\Phi(z)) &= \tfrac{1}{2} \textstyle\sum_{i=1}^{n} \frac{z_i^2}{\sigma_i(z_i)} \\ L_o(\Phi(z)) &= \tfrac{1}{2} \textstyle\sum_{i=1}^{n} z_i^2 \sigma_i(z_i). \end{aligned}$$

In particular, if $U = \mathbb{R}^n$, then

$$\|\Sigma\|_H = \sup_{z_1 \in \mathbb{R}} \sigma_1(z_1).$$

Proof: Suppose that the system is already balanced in the sense of Theorem 4 without loss of generality. Theorem can be proved by just defining the new singular value functions

$$\sigma_i(z_i) := \rho_i(\phi_i(z_i))$$

with a coordinate transformation $x = \Phi(z) = (\phi_1(z_1), \phi_2(z_1), \ldots, \phi_n(z_n))$ where $z_i = \phi_i^{-1}(x_i) := x_i \sqrt{\rho_i(x_i)}$. ■

We call the state-space realization obtained in Theorem 5 by *balanced realization* of the system (4). In fact, the equations can be rewritten as

$$L_c(\Phi(z)) = \frac{1}{2} z^{\mathrm{T}} P(z)^{-1} z, \quad L_o(\Phi(z)) = \frac{1}{2} z^{\mathrm{T}} Q(z) z$$

$$P(z) = Q(z) = \mathrm{diag}(\sigma_1(z_1), \sigma_2(z_2), \ldots, \sigma_n(z_n))$$

which is quite a natural generalization of the balanced realization in the linear case (3). The functions σ_i's and ρ_i's are essentially the same and both of them are the singular values of the Hankel operator $\mathcal{H}$ indeed.

V. Conclusion

This paper was devoted to the new characterization of the balanced realization of nonlinear dynamical systems based on singular value analysis of Hankel operators. It has been proved that it is always possible to let the coordinate axes of the state-space appear separately in the controllability and observability functions. This fact can be utilized to derive a *real* balanced realization containing the balancing between the input-to-state behavior and the state-to-output behavior. The authors believe that the proposed result will provide a new basis for model reduction and system identification of nonlinear systems.

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